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Integrability of two coupled oscillators using complex coordinates

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Abstract A general method is used for finding the second constant of motion of fourth order in complex coordinates for two dimensional classical systems A general potential equation is obtained which provides a large class of second constant of motion for two dimensional systems However, it has been observed that the arbitrary constants are chosen in such a way that system becomes exactly integrable

Keywords Classical invariants, integrable systems, second constant of motion

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Whittaker [1] first investigated the problem of the construction of an invariant other than the total energy which will be called the second constant of motion for a system

$$\ddot{x}_1 = -\frac{\partial V}{\partial x_1} \quad \ddot{x}_2 = -\frac{\partial V}{\partial x_2},$$

where $V = V(x_1, x_2)$.

His studies were however restricted to the invariant of first or second order in momenta. In recent years, although there have been several attempts [1-9] made to construct the second order invariants, not much efforts have been made to obtaining the higher order invariants for such systems.

The utility of the second or higher order constant of motion, if they can be constructed for a system, it has been noticed that it may reduce some nonlinear dynamical problems to a quadrature. In fact, there already exists scarcity of classically integrable systems in higher degrees of freedom.

By introducing the complex variable $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, some simplifications [5] were achieved in the derivation and analysis which turn out to be more transparent. We have examined the time-independent systems for the potential $V = lz^4 + mz^2\bar{z}^2 + n\bar{z}^4$ in two dimensions.

A simple analysis reveals an interesting case of two coupled oscillators which somehow could not be explicitly investigated earlier using complex coordinates. Here, we derive a general potential equation and construct the second constant of motion for the said potential where l, m, n are arbitrary constants. Using rationalisation method, we obtain $l = n = 1$, $m = 6$, which is a case of two coupled quartic oscillators.

We consider the dynamical system whose Lagrangian is

$$L = \frac{1}{2}|\dot{z}|^2 - V(z, \bar{z}), \tag{1}$$

with the concomitant equations of motion

$$\ddot{z} = -2\frac{\partial V}{\partial \bar{z}}, \ \ddot{\bar{z}} = -2\frac{\partial V}{\partial z}. \tag{2}$$

Assuming the existence of the second constant of motion upto fourth order in momenta in the form of

$$I = a_0 + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{4!}a_{ijkl}\,\xi_i\xi_j\xi_k\xi_l, \tag{3}$$

where $i, j, k, l = 1, 2$, $\xi_1 = \dot{z}$, $\xi_2 = \dot{\bar{z}}$ and a_0, a_{ij}, a_{ijkl} are functions of $z, \bar{z}$.

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The invariant I implies $dI/dt = 0$ and using (3) we get

$$a_{0,i}\xi_i + \frac{1}{2}a_{ij,k}\xi_i\xi_j\xi_k + \frac{1}{2}a_{ij}\left(\dot{\xi}_i\xi_j + \xi_i\dot{\xi}_j\right) + \frac{1}{24}a_{ijkl,m}\xi_i\xi_j\xi_k\xi_l\xi_m$$

$$+\frac{1}{24}a_{ijkl}\left(\dot{\xi}_i\xi_j\xi_k\xi_l + \xi_i\dot{\xi}_j\xi_k\xi_l + \xi_i\xi_j\dot{\xi}_k\xi_l + \xi_i\xi_j\xi_k\dot{\xi}_l\right) = 0 \quad (4)$$

After accounting for the proper symmetrisation of the coefficients and since (4) must hold identically in $\xi_i's$, we can obtain the following relations .

$$a_{ijkl,m} + a_{jklm,i} + a_{klmi,j} + a_{lmij,k} + a_{mijk,l} = 0, \quad (5)$$

$$a_{ij,k} + a_{jk,i} + a_{ki,j} + a_{ijkl}\dot{\xi}_l = 0, \quad (6)$$

$$a_{0,i} + a_{ij}\dot{\xi}_j = 0. \quad (7)$$

Eqs (5)-(7) yield the following set of partial differential equations

$$\frac{\partial a_{1111}}{\partial z} = 0, \quad (8)$$

$$\frac{\partial a_{1111}}{\partial \bar{z}} + 4\frac{\partial a_{1112}}{\partial z} = 0, \quad (9)$$

$$3\frac{\partial a_{1122}}{\partial z} + 2\frac{\partial a_{1112}}{\partial \bar{z}} = 0, \quad (10)$$

$$3\frac{\partial a_{1122}}{\partial \bar{z}} + 2\frac{\partial a_{1222}}{\partial z} = 0, \quad (11)$$

$$4\frac{\partial a_{1222}}{\partial \bar{z}} + \frac{\partial a_{2222}}{\partial z} = 0, \quad (12)$$

$$\frac{\partial a_{2222}}{\partial \bar{z}} = 0, \quad (13)$$

$$3\frac{\partial a_{11}}{\partial z} = 2a_{1111}\frac{\partial V}{\partial z} + 2a_{1112}\frac{\partial V}{\partial z}, \quad (14)$$

$$3\frac{\partial a_{22}}{\partial \bar{z}} = 2a_{2222}\frac{\partial V}{\partial z} + 2a_{1222}\frac{\partial V}{\partial z}. \quad (15)$$

$$\frac{\partial a_{11}}{\partial \bar{z}} + 2\frac{\partial a_{12}}{\partial z} = 2a_{1112}\frac{\partial V}{\partial z} + 2a_{1122}\frac{\partial V}{\partial \bar{z}} \quad (16)$$

$$\frac{\partial a_{22}}{\partial z} + 2\frac{\partial a_{12}}{\partial \bar{z}} = 2a_{1222}\frac{\partial V}{\partial \bar{z}} + 2a_{1122}\frac{\partial V}{\partial \bar{z}}, \quad (17)$$

$$\frac{\partial a_0}{\partial} = 2a_{11}\frac{\partial V}{\partial \bar{z}} + 2a_{12}\frac{\partial V}{\partial \bar{z}} \quad (18)$$

$$\frac{\partial a_0}{\partial \bar{z}} = 2a_{12}\frac{\partial V}{\partial \bar{z}} + 2a_{22}\frac{\partial V}{\partial \bar{z}}. \quad (19)$$

To solve these equations, we notice from (8) and (13), we have $a_{1111} = a_{1111}(\bar{z}) = \sigma_1(\bar{z})$ and $a_{2222} = a_{2222}(z) = \chi_1(z)$ Now differentiating eq. (9) with respect to z to eliminate a_{1111} using eq. (8) we have $\partial^2 a_{1112}/\partial z^2 = 0$, which admits the solution as

$$a_{1112} = \sigma_2(\bar{z})z + \sigma_3(\bar{z}). \quad (20)$$

Similarly, eqs. (12) and (13) will lead to the solution as

$$a_{2222} = \chi_2(z)\bar{z} + \chi_3(z). \quad (21)$$

On differentiating eq. (10) with respect to $\bar{z}$ and correspondingly subtracting the results and making use of eq (20) and (21), we obtain

$$\frac{d^2\sigma_2}{d\bar{z}^2} + \frac{d^2\sigma_3}{d\bar{z}^2} = \bar{z}\frac{d^2\chi_2}{dz^2} + \frac{d^2\chi_3}{dz^2}$$

Now, we fix $\sigma_3 = D_1$ and $\chi_3 = D_2$, the above equation reduces to the form

$$\frac{1}{\bar{z}}\frac{d^2\sigma_2}{d\bar{z}^2} - \frac{1}{z}\frac{d^2\chi_2}{dz^2} = \text{Constant}\ (D_3),$$

$$\sigma_2 = (1/6)D_3\bar{z}^3 + D_4\bar{z} + D_5,$$

$$\chi_2 = (1/6)D_3 z^3 + D_6 z + D_7.$$

Substituting these values of $\sigma_2, \chi_2, \sigma_3$ and χ_3 in eqs. (20) and (21), we have

$$a_{1112} = \frac{1}{6}D_3 z\bar{z}^3 + D_4 z\bar{z} + D_5 z + D_1, \quad (22)$$

$$a_{1222} = \frac{1}{6}D_3 z^3\bar{z} + D_6 z\bar{z} + D_7\bar{z} + D_2 \quad (23)$$

Similarly, we find σ_1 and χ_1 starting from eq. (9) and (12) as

$$\sigma_1 = a_{1111} = -\frac{1}{6}D_3\bar{z}^4 - 2D_4\bar{z}^2 - 4D_5\bar{z} + D_8,$$

$$\chi_1 = a_{2222} = -\frac{1}{6}D_3 z^4 - 2D_6 z^2 - 4D_7 z + D_9.$$

Using eqs. (22) and (23) in eqs. (10) and (11), we obtain

$$\frac{\partial a_{1122}}{\partial z} = -(2/3)z\frac{d\sigma_2}{\partial \bar{z}} = -(1/3)D_3 z\bar{z}^2 - (2/3)D_4 z$$

or $a_{1122} = -(1/6)D_3 z^2\bar{z}^2 - (1/3)D_4 z^2 - (1/3)D_6\bar{z}^2 - +D_{10}$.

Finally, the solution of eqs. (8)-(13) yield

$$a_{1111} = -(1/6)D_3\bar{z}^4 - 2D_4\bar{z}^2 - 4D_5\bar{z} + D_8, \quad (24)$$

$$a_{1112} = (1/6)D_3 z\bar{z}^3 + D_4 z\bar{z} + D_5 z + D_1, \quad (25)$$

$$a_{1122} = -(1/6)D_3 z^2\bar{z}^2 - (1/3)D_4 z^2 - (1/3)D_6\bar{z}^2 + D_{10}, \quad (26)$$

$$a_{1222} = (1/6)D_3 z^3 \bar{z} + D_6 z\bar{z} + D_7 \bar{z} + D_2, \tag{27}$$

$$a_{2222} = -(1/6)D_3 z^4 - 2D_6 z^2 - 4D_7 z + D_9 . \tag{28}$$

In order to eliminate a_{11}, a_{22} and a_{12} from eqs. (14)-(17), we differentiate equation (16) with respect to z and making use of (14) for $\partial a_{11} / \partial z$ we obtain

$$\frac{\partial^2 a_{12}}{\partial z^2} = \frac{\partial}{\partial z}\left(a_{1112}\frac{\partial V}{\partial z} + a_{1122}\frac{\partial V}{\partial \bar{z}} \right.$$

$$\left. -\frac{1}{3}\frac{\partial}{\partial \bar{z}}\right| a_{1111}\frac{\partial V}{\partial z} + a_{1112}\frac{\partial V}{\partial z} \tag{29}$$

Similarly, on differentiating (17) with respect to $\bar{z}$ and making use of (15) for $\partial a_{22} / \partial \bar{z}$, we obtain

$$\frac{\partial^2 a_{12}}{\partial \bar{z}^2} = \frac{\partial}{\partial z}\left(a_{1222}\frac{\partial V}{\partial \bar{z}} + a_{1122}\frac{\partial V}{\partial z} \right.$$

$$\left. \frac{1}{3}\frac{\partial}{\partial z}\left(a_{2222}\frac{\partial V}{\partial \bar{z}} + a_{1222}\frac{\partial V}{\partial z} \right)\right. . \tag{30}$$

Now to eliminate a_{12}, we again differentiate twice eq. (29) with respect to $\bar{z}$ and equation (30) with respect to z using $\partial^4 a_{12} / \partial z^2 \partial \bar{z}^2 = \partial^4 a_{12} / \partial \bar{z}^2 \partial z^2$, we finally obtain

$$\left(\frac{\partial^3 a_{1122}}{\partial z \partial \bar{z}^2} - \frac{1}{3}\frac{\partial^3 a_{1112}}{\partial \bar{z}^3} - \frac{\partial^3 a_{1222}}{\partial \bar{z}\partial z^2} + \frac{1}{3}\frac{\partial^3 a_{2222}}{\partial z^3} \right)\frac{\partial V}{\partial z}$$

$$+\left(\frac{\partial^2 a_{1122}}{\partial \bar{z}^2} - \frac{\partial^2 a_{1222}}{\partial z \partial \bar{z}} + \frac{\partial^2 a_{2222}}{\partial z^2} \right)\frac{\partial^2 V}{\partial z^2} + \left(\frac{\partial a_{2222}}{\partial z} - \frac{\partial a_{1222}}{\partial \bar{z}} \right)\frac{\partial^3 V}{\partial z^3}$$

$$+\frac{1}{3}a_{2222}\frac{\partial^4 V}{\partial z^4} + \left(\frac{\partial a_{1122}}{\partial \bar{z}} - \frac{\partial a_{1222}}{\partial z} \right)\frac{\partial^3 V}{\partial \bar{z}\partial z^2} - \frac{2}{3}a_{1222}\frac{\partial^4 V}{\partial \bar{z}\partial z^3}$$

$$+\frac{2}{3}a_{1112}\frac{\partial^4 V}{\partial z \partial \bar{z}^3} + \left(\frac{\partial a_{1112}}{\partial z} - \frac{\partial a_{1122}}{\partial z} \right)\frac{\partial^3 V}{\partial z \partial \bar{z}^2}$$

$$+\left(\frac{\partial^3 a_{1112}}{\partial z \partial \bar{z}^2} + \frac{1}{3}\frac{\partial^3 a_{1222}}{\partial z^3} - \frac{\partial^3 a_{1122}}{\partial \bar{z}\partial z^2} - \frac{1}{3}\frac{\partial^3 a_{1111}}{\partial \bar{z}^3} \right)\frac{\partial V}{\partial \bar{z}}$$

$$+\left(\frac{\partial^2 a_{1112}}{\partial z \partial \bar{z}} - \frac{\partial^2 a_{1122}}{\partial z^2} - \frac{\partial^2 a_{1111}}{\partial \bar{z}^2} \right)\frac{\partial^2 V}{\partial \bar{z}^2}$$

$$\frac{\partial a_{1112}}{\partial z} \quad \frac{\partial a_{1}}{\partial \bar{z}} \Big) \frac{\partial^3 V}{\partial \bar{z}^3} - \frac{1}{3}a_{1111}\frac{\partial^4 V}{\partial \bar{z}^4} = 0. \tag{31}$$

This is a potential equation corresponding to the fourth-order invariant which involves the potential derivatives and known coefficients a_{ijkl} through unknown constant D_i's. On substituting these coefficients, the potential equation reduces to a simpler form

$$-\frac{10}{3}D_3 z\frac{\partial V}{\partial z} - \frac{17}{3}\left(\frac{1}{2}D_3 z^2 + D_6\right)\frac{\partial^2 V}{\partial z^2} - 5\left(\frac{1}{6}D_3 z^3 + D_6 z + D_7\right)$$

$$\cdot\frac{\partial^3 V}{\partial z^3} + \frac{1}{3}\left(-\frac{1}{6}D_3 z^4 - 2D_6 z^2 - 4D_7 z + D_9\right)\frac{\partial^4 V}{\partial z^4}$$

$$-\frac{5}{3}\left(\frac{1}{2}D_3 z^2 z + D_6\bar{z}\right)\frac{\partial^3 V}{\partial \bar{z}\partial z^2} - \frac{2}{3}\left(\frac{1}{6}D_3 z^3\bar{z} + D_6 z\bar{z} + D_7\bar{z} + D_2\right)$$

$$\times\frac{\partial^4 V}{\partial z \partial z^3} + \frac{5}{3}\left(\frac{1}{2}D_3 z\bar{z}^2 + D_4 z\right)\frac{\partial^3 V}{\partial z \partial \bar{z}^2}$$

$$+\frac{2}{3}\left(\frac{1}{6}D_3 z\bar{z}^3 + D_4 zz + D_5 z + D_1\right)\frac{\partial^4 V}{\partial z \cdot \partial \bar{z}^3} + \frac{10}{3}D_3\bar{z}\frac{\partial V}{\partial \bar{z}}$$

$$+\frac{17}{3}\left(\frac{1}{2}D_3\bar{z}^2 + D_4\right)\frac{\partial^2 V}{\partial \bar{z}^2} + 5\left(\frac{1}{6}D_3 z^3 + D_4\bar{z} + D_5\right)\frac{\partial^3 V}{\partial \bar{z}^3}$$

$$+\frac{1}{3}\left(\frac{1}{6}D_3\bar{z}^4 + 2D_4\bar{z}^2 + 4D_5\bar{z} - D_8\right)\frac{\partial^4 V}{\partial \bar{z}^4} = 0 . \tag{32}$$

For a given form of potential $V(z, \bar{z})$, the unknown constants D_i's can be determined by rationalising the potential equation, subsequently, the determination of other coefficients a_0, a_{ij} can be obtained which would lead to the final form of the second constant of motion.

We consider the potential

$$V(z, \bar{z}) = lz^4 + mz^2\bar{z}^2 + n\bar{z}^4 . \tag{33}$$

On substituting the derivatives of this potential in eq. (32) and rationalising the resultant equation, we find that $l = n = 1$ and $m = 6$, $D_3 = D_4 = D_6 = D_7 = 0$. For this potential, the value of a_{11} and a_{22} is obtained from eqs. (14) and (15) respectively

$$a_{11} = \frac{2}{3}D_1 z^4 + \frac{8}{3}D_8\bar{z}z^3 + 4D_1\bar{z}^2 z^2 + \frac{8}{3}D_8\bar{z}^3 z + \frac{2}{3}D_1\bar{z}^4 , \tag{34}$$

$$a_{22} = \frac{2}{3}D_2\bar{z}^4 + \frac{8}{3}D_9 z\bar{z}^3 + 4D_2 z^2\bar{z}^2 + \frac{8}{3}D_9 z^3\bar{z} + \frac{2}{3}D_2 z^4 \tag{35}$$

and the value of a_{12} is obtained from eqs. (16) and (17) and putting $D_1 = D_2$, $D_8 = D_9$ as

$$a_{12} = -\frac{1}{3}D_9 z^4 + D_{10}z^4 + \frac{8}{3}D_2\bar{z}z^3 + 6D_{10}z^2\bar{z}^2 - 2D_9 z^2\bar{z}^2$$

$$+\frac{8}{3}D_2\bar{z}^3 z - \frac{1}{3}D_9\bar{z}^4 + D_{10}\bar{z}^4 . \tag{36}$$

We obtain the value of a_0 using eqs. (18) and (19) as

$$a_0 = D_{10}\bar{z}^8 - \frac{1}{3}D_9\bar{z}^8 + \frac{16}{3}D_2\bar{z}^7 z + \frac{20}{3}D_9 z^2\bar{z}^6 + 12D_{10}z^2\bar{z}^6$$

$$+\frac{112}{3}D_2\bar{z}^5z^3+38D_{10}z^4\bar{z}^4+\frac{26}{3}D_9z^4\bar{z}^4+\frac{112}{3}D_2\bar{z}^3z^5$$

$$+\frac{20}{3}D_9z^6\bar{z}^2+12D_{10}z^6\bar{z}^2+\frac{16}{3}D_2\bar{z}z^7+D_{10}z^8-\frac{1}{3}D_9z^8. \quad (37)$$

Now substituting the value of a_0, a_{11}, a_{22} and a_{12} in eq. (3), the corresponding invariant becomes

$$I=D_{10}\left(z^8+\bar{z}^8\right)-\frac{1}{3}D_9\left(z^8+\bar{z}^8\right)+\frac{16}{3}D_2\left(\bar{z}z^7+\bar{z}^7z\right)$$

$$\frac{20}{3}D_9\left(z^2\bar{z}^6+z^6\bar{z}^2\right)+12D_{10}\left(z^2\bar{z}^6+z^6\bar{z}^2\right)$$

$$+\frac{112}{3}D_2\left(\bar{z}^3z^5+\bar{z}^5z^3\right)+38D_{10}z^4\bar{z}^4+\frac{26}{3}D_9z^4\bar{z}^4$$

$$\frac{1}{}D_2\bar{z}^4+\frac{4}{}D_9\bar{z}^3z+2D_2\bar{z}^2z^2+\frac{4}{}D_9\bar{z}z^3+\frac{1}{}D_2z^4$$

$$\times\left(\dot{z}^2+\dot{\bar{z}}^2\right)+\left(-\frac{1}{3}D_9z^4+D_{10}z^4+\frac{8}{3}D_2\bar{z}z^3+6D_{10}z^2\bar{z}^2\right.$$

$$\left.-2D_9z^2\bar{z}^2+\frac{8}{3}D_2\bar{z}^3z-\frac{1}{3}D_9\bar{z}^4+D_{10}\bar{z}^4\right)\dot{z}\dot{\bar{z}}+\frac{1}{24}D_9\dot{z}^4$$

$$+\frac{1}{24}D_9\dot{\bar{z}}^4+\frac{1}{6}D_2\dot{z}^3\dot{\bar{z}}+\frac{1}{4}D_{10}z^2\bar{z}^2+\frac{1}{6}D_2z\dot{\bar{z}}^3. \quad (38)$$

Here, we have studied the classical integrable systems using complex coordinates. We establish a general potential equation of fourth order. Rationalising the potential eq. (32), one can construct the second constant of motion. Interestingly considering a potential of the type $V(z,\bar{z})=lz^4+mz^2\bar{z}^2+n\bar{z}^4$, we find the relation between the arbitrary constants l, m and n as $l=n=1$, $m=6$ which satisfy for the construction of the second constant of motion (38).

The usefulness of the potential equation (32) would provide a large class of potentials whose second constant of motion can be constructed. One can also construct the second constant of motion using generalised coordinates [10].

The utility of the second constants of motion if constructed for a system, has been noticed in recent years from several points of view, particularly solving several nonlinear problems of plasma physics and hydrodynamics, and also in the study of a classical analog of Yang-Mills field equations with reference to the generation of potentials for both time-independent and time-dependent by choosing the suitable gauges. This invariant also helps us to find a general class of soliton solutions where some integral constraints on the fields exist which are, in turn, the consequences of appropriate physical conservation laws, such as charge, iso-spin *etc* [11]. The higher-order invariants determine the internal symmetry of a physical system particularly in molecular dynamics [12].

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